STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

TW 80

Topological foundations of the theory of distributions

by

H. A. Lauwerier

Part I

October 1961

Index

The following abbreviations are used:
l.t.s. = linear topological space, n.l.s. = normed linear space, s.n.s. = sequentially normed space, u.s.n.s. = union of sequentially normed spaces, c.s. = conjugate space.

Some important theorems

0. Introduction

In the following ten sections we have tried to give a rather complete and sufficiently detailed treatment of the main notions and concepts of functional analysis which are indispensable for a clear understanding of the theory of generalized functions or distributions. The material is mainly derived from the first chapter of Gel'fand and Shilov, Generalized functions II (Moscow 1958, in Russian). The material of the first few introductory sections is derived from Kolmogorov and Fomin, Elements of the theory of functions and functional analysis (Roch. 1957) and Ljusternik and Sobolew, Elemente der Funktionalanalysis (Berlin 1955, in German). At a few places we have also consulted A.E. Taylor, Introduction to Functional analysis (New York 1957) and the encyclopaedic work of Dunford and Schwartz, Linear operators, vol. I (New York 1958).

The following treatment is written for the general reader. This means that he needs no previous knowledge of topology or functional analysis, nor need he consult the textbooks on the subject. However, those who are not very familiar with the language of functional analysis are strongly advised to read the little book of Kolmogorov and Fomin which is a very clearly written introduction to the subject.

The topological treatment of generalized functions was originally given by L. Schwartz in a very elaborate fashion in his two books on distributions. However, they make hard reading for the uninitiated who is not a "Bourbakist". On the other hand, the approach of Gel'fand and Shilov has made it possible to present the following self-contained treatment which may be followed without being involved in too many topological details. The main emphasis is on the concept of a sequentially normed space which is somewhere between the too general linear topological space and the too special linear normed space.

A very important special type of sequentially normed space is obtained by taking the collection of all real functions $\varphi(x)$ of a real variable $-\infty < x < \infty$ which are infinitely differentiable and which vanish outside the fixed interval $(-a,a)$. This space will be indicated by $K(a)$. In $K(a)$ a sequence of norms is introduced by

$$\|\varphi(x)\|_1 = \max|\varphi(x)| \ , \ \|\varphi(x)\|_2 = \max\{|\varphi(x)|, |\varphi'(x)|\},\ldots,$$
$$\|\varphi(x)\|_m = \max\{|\varphi(x)|, |\varphi'(x)|,\ldots, |\varphi^{(m-1)}(x)|\}\ldots.$$

Next a suitable topology is introduced which implies that convergence to zero of a sequence $\varphi_n(x)$ $(n=1,2,\ldots)$ means that uniformly in $(-a,a)$

$$\varphi_n(x)\to 0, \ \varphi_n'(x)\to 0,\ldots, \varphi_n^{(m)}(x)\to 0,\ldots$$

In this way K(a) becomes a special type of linear topological space which does not have the topology of a Banach space but which has many properties in common with the latter type of space.

Generalized functions or distributions of a single real variable may be defined as continuous linear functionals (f, φ) on the space K which is the union of all spaces K(a) $(a\to\infty)$. Many properties of generalized functions are reflections of the topological structure of the sequentially normed spaces K(a).

Since there is a great variety of spaces like K(a) and K on which generalized functions can be defined, an abstract approach by means of functional analysis is of paramount importance. It can be said that the concept of sequentially normed spaces and their unions covers all important cases.

The set-up of the treatment given here is as follows. Sections 1 and 2 are introductory. In section 1 the definition of a linear space is given. In section 2 we consider a topological space with a suitable separation axiom. Section 3 on linear topological spaces combines the results of the preceding two sections. This section is of fundamental importance for the remainder. We note the definitions of boundedness, of a continuous linear operator and a continuous linear functional. It is shown that in the space of all continuous linear functionals a strong topology and a weak topology can be introduced. In section 4 we consider a metric space and in particular a linear metric space, which is a special kind of topological space. The most important theorems here are those of Baire (theorem 4.1) and of Banach (theorem 4.6). Further, in view of later applications in section 9, something is said about compactness and a proof of the well-known theorem of Arzelá-Ascoli is given (theorem 4.4). In section 5 we consider the more familiar normed spaces. Some properties are derived from those of a linear topological space and a linear metric space by making suitable specializations. It is shown that to each continuous linear operator and in particular to each continuous linear functional we may associate a norm in which the strong topology finds a simple expression. In this section we meet the so-called principle of uniform boundedness which is of fundamental importance. The main theorem is that of Banach and Steinhaus (theorem 5.3). We note that this principle rests upon Baire's theorem of the preceding section. As another important principle we mention the theorem of Hahn-Banach (theorem 5.6) on the continuation of a continuous linear functional. In view of later applications a proof is given of the representation theorem of Riesz (theorem 5.10) on a continuous linear functional on the space $C(0,1)$ of all continuous functions $\varphi(x)$, $0 \leqq x \leqq 1$. In section 6 the definition of a sequentially normed space is given and it is shown that a s.n.s. may be considered as a linear metric space with an equivalent topology. This fact makes it possible to apply e.g. Baire's theorem which leads to the useful lemma of theorem 6.2. Further a proof is given of the theoretically important fact that the topology of a s.n.s. is essentially different from that of a normed space. The subject is continued in section 7 where in particular the properties of continuous linear functionals on a s.n.s. are studied. Many properties are either specializations of more general properties for a linear topological space or generalizations of more special properties for a normed space. The principle of uniform boundedness appears here in the theorems 7.2, 7.4 and 7.7. We note in particular that in the conjugate space of a s.n.s. the notions weakly bounded and strongly bounded are equivalent. We mention also the important fact (theorem 7.5) that the conjugate space is complete with respect to weak convergence. In this section it is shown by means of Riesz' representation theorem that an explicit form can be obtained for continuous linear functionals on the space $K(a)$. In section 8 something is said about linear operators mapping a s.n.s. X into a similar one X^*. Also for continuous linear operators a convergence in the weak sense can be introduced. It is shown that the space of continuous linear operators $T(X \longrightarrow X^*)$ is complete with respect to weak convergence (theorem 8.4). In section 9 we discuss perfect spaces, which are sequentially normed spaces with compact bounded subsets. Perfect spaces enjoy a number of nice properties. Theorem 9.2 says that in the conjugate of a perfect space weak and strong convergence are equivalent. Further a perfect space is separable (theorem 9.5). In the conjugate of a perfect space bounded sets are compact as in the original space (theorem 9.4). It is shown that in particular $K(a)$ is a perfect space. The last section does not contain essentially new features. A definition is given of the union $X^{(\omega)}$ of a sequence of sequentially normed spaces $X^{(m)}$. The properties of the union are always reduced to those of its constituting members. In the union $X^{(\omega)}$ no special topology is introduced but we restrict ourselves to the introduction of a weak convergence. As an obvious generalization of a former result it is shown that the conjugate $X^{(\omega)'}$ is complete with respect to this (weak) convergence. This section closes with a few remarks on linear operators.

Contents

1. Linear spaces

A set X of elements $x,y,z,\dots$ is said to be a <u>linear space</u> if the following conditions are satisfied:

<u>I</u>. X is an additive Abelian group.
This means that for any two elements $x,y \in X$ there is uniquely defined a third element $z \in X$, called their sum and written as $z=x+y$. The addition is commutative and associative, there is a zero element 0 [1] and each element x has an inverse $-x$. Formally we have the following group of axioms:

I 1° $x+y=y+x$ for each x and y.
I 2° $x+(y+z)=(x+y)+z$ for each x,y and z.
I 3° $x+0=x$ for each x.
For each x there is a $-x$ such that
I 4° $x+(-x)=0$.

These axioms imply that the zero element is unique. Further for each element the inverse is uniquely determined.

II. For each element $x \in X$ and each real number α there is uniquely defined a second element of X, called the scalar product and written as αx. For the scalar multiplication there are two axioms

II 1° $\alpha(\beta x) = (\alpha\beta)x$.
II 2° $1\cdot x = x$.

III. Addition and scalar multiplication are distributive with respect to each other.

III 1° $\alpha(x+y) = \alpha x+\alpha y$.
III 2° $(\alpha+\beta)x = \alpha x+\beta x$.

From these axioms the following rules can easily be derived.

(1.1) $$(-1)\cdot x = -x.$$

(1.2) $$\alpha\cdot 0 = 0.$$

(1.3) $$0\cdot x = 0.$$

(1.4) $$\alpha x = 0 \text{ implies } \alpha=0 \text{ or } x=0.$$

The set of all vectors (or points) in Euclidean 3-space forms a special case of a linear space. An abstract linear space embodies so many features of this particular case that words like vector have been taken over into a more general context. Thus a linear space is often called <u>a vector space</u> and the elements are called vectors.

1) The fact that the zero element of X and the real number zero are indicated by the same symbol should not cause confusion.

In the foregoing it was assumed that the scalars $\alpha, \beta, \ldots$ were real numbers. To emphasize this the space is sometimes called a real linear space. With the same set of axioms it is possible to define a somewhat more general type of linear space for scalars belonging to some commutative field. However, we confine ourselves to the two fields of real and complex numbers, respectively.

A subset M of X is called a linear manifold if with x and y any linear combination $\alpha x+\beta y$ belongs to M. It will be seen at once that a linear manifold is itself a linear space. Therefore it may be called a subspace.

A finite set of vectors $x_1, x_2, \ldots, x_m$ is said to be linearly dependent if there exists a set of scalars $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_m x_m=0. \tag{1.5}$$

If the vectors $x_1, x_2, \ldots, x_m$ are not linearly dependent the relation (1.5) would imply $\alpha_1=\alpha_2=\ldots=\alpha_m=0$. In that case the vectors are said to be linearly independent. An infinite set of vectors is called linearly independent if every finite subset is linearly independent.

If X contains n linearly independent vectors but if every set of n+1 vectors is linearly dependent the space X is said to be finite dimensional with the dimension n. Otherwise the space is said to be infinite dimensional. As we shall see the spaces of greatest interest for us are infinite dimensional. If X has a finite dimensional subspace M with the dimension n any set of n linearly independent vectors $x_1, x_2, \ldots, x_n$ is said to be a base of M. Any element $x \in M$ can be expressed as a linear combination

$$x = \xi_1 x_1+\xi_2 x_2+\ldots+\xi_n x_n. \tag{1.6}$$

A linear operator is defined as a transformation T which transforms the elements x,y,... of a linear space X into elements Tx,Ty,... of a linear space X_1 and for which

$$T(\alpha x+\beta y) = \alpha Tx +\beta Ty.$$

The points $x \in X$, upon which the operator can be applied, constitute the domain of T. The points $x_1 \in X_1$ which are obtained as $x_1=Tx$ constitute the range of T. Hence the domain is all of X and the range is a part or all of X_1. Obviously the range of T is a subspace of X_1. If Tx=Ty implies x=y the operator defines a one-to-one mapping between two linear spaces so that T has an inverse T^{-1}. Obviously T^{-1} is also linear. It follows easily from the linearity of T that an inverse of T exists if and only if $Tx=0$ implies $x=0$.

For linear operators $T(X \to X_1)$ an addition and a scalar multiplication can be defined by

$$(T_1+T_2)x = T_1x+T_2x,$$
$$(\alpha T)x = \alpha(Tx).$$

If we define the zero operator as the transformation which maps all elements of X into the zero of X_1 it is easily seen that the set of all linear operators $T(X \to X_1)$ is itself a linear space.

We shall now introduce a number of definitions and notations which will be frequently used in the subsequent sections.

If S is a subset of the linear space X then the set of all elements αx for which $x \in S$ is written as αS. In case $\alpha=-1$ we write $-S$ for $(-1)S$. If R and S are subsets of X then the set of all elements $x \pm y$ for which $x \in R$ and $y \in S$ is written as $R \pm S$. We note that in general $2S \neq S+S$.

A subset S of X is said to be symmetric if $-S=S$.

A subset S of X is said to be convex if $\alpha S+(1-\alpha)S=S$ for all α with $0 \leq \alpha \leq 1$.

A subset S of X is said to be balanced if $\alpha S \subset S$ for all α with $|\alpha| \leq 1$.

A subset S of X is said to be absorbing if for each element $x \in X$ there is a scalar $\alpha \neq 0$ with $x \in \alpha S$.

2. Topological spaces

A set X of elements x,y,z,... is said to be a topological space if there is a family of special subsets called open sets which satisfy the following axioms

1° The empty set and the whole space are open.

2° The union of open sets, even uncountably many, is open.

3° The intersection of a finite number of open sets is open.

The family of open sets is sometimes called a topology for X. Sometimes we consider several topologies for the same space X. If we have two topologies τ_1 and τ_2 for X and if every member of τ_1 is also a member of τ_2 we say that the topology τ_1 is weaker than τ_2 or that τ_2 is stronger than τ_1. Thus the stronger topology has more open sets.

A set S of a topological space is called closed if its complement S' is open. It follows from the axioms that the empty set and the whole space are also closed. Further, that the intersection of a collection of closed sets is again closed, and that the union of a finite number of closed sets is again closed.

Any open set which contains a point x is called a neighbourhood of x. A point x is called a limit point or accumulation point of a set S if every neighbourhood of x contains a point of S distinct from x. Clearly a closed set may be characterized by the property of containing all its limit points. The closed set which is obtained from an arbitrary set S by addition of all its limit points is called the closure of S and is denoted by $\bar{S}$.

To the axioms 1° 2° 3° we add a fourth axiom of a special nature and which is called a separation axiom

4° Each set consisting of a single point is closed.

This axiom can be replaced by other related axioms which may lead to topological spaces of a slightly less general type. We shall, however, not enter into this question. The separation axiom stated above has the following simple but important consequence.

If x_1 and x_2 are distinct points there exists a neighbourhood of x_1 which does not contain x_2. In fact, the set which is the complement of the single point x_2 is open and contains x_1.

A system B of neighbourhoods U of x is said to be a base at x if each neighbourhood of x contains a member of B.

Example

Consider the points of a Euclidean plane. A base at the origin (0,0) may be formed by the open circles $x^2+y^2 < n^{-2}$ (n=1,2,...).

A topological space which has a countable base at x is said to satisfy the first axiom of countability at x. In that case we may suppose that the neighbourhoods U_n (n=1,2,...) are ordered in such a way that $U_1 \supset U_2 \supset U_3 \supset \dots$. Then the base is called a shrinking base.

The set S is said to be dense in X if its closure is X. If there exists a countable dense set in X the space X is said to be separable.

Example

X consists of the real points x in (0,1). The rational points form a countable set which is dense in X. Hence X is separable.

The sequence x_n (n=1,2,...) is said to converge to the limit x,

$$\lim x_n = x,$$

if for each neighbourhood U(x) of x there is an index N such that $x_n \in U(x)$ for $n > N$.

If x is the limit point of the set S and if X has a countable base at x then it is possible to select a sequence x_n of elements of S for which $\lim x_n = x$. The proof is simple. However, the proposition may be not true if X does not satisfy the first countability axiom.

Example

X consists of all real bounded functions f(x) defined on the interval (0,1). A neighbourhood of $f_0(x)$ is obtained by taking all f(x) which for a given positive ε satisfy

$$|f(x)-f_0(x)| < \varepsilon$$

at a finite number of points, say $x_1, x_2, \dots, x_m$.
We consider the set S of those functions g(x) for which $g(x) \equiv 1$ with a finite number of exceptions where g(x)=0. The function $g(x) \equiv 0$ is clearly a limit point of S. However, it is not possible to select a sequence $g_n \in S$ which converges to 0.

The set S is said to be (sequentially) compact in X if each infinite subset of S contains a converging sequence of distinct elements. We note that in this definition the limit of the sequence does not necessarily belong to S.

The concept of an operator and its inverse can be introduced as in the previous section.
The operator A which transforms the points x of a topological space X into the points x_1 of a topological space X_1 is said to be <u>continuous</u> if for any neighbourhood U_1 of x_1 there is a neighbourhood U of x the image of which is in U_1.

3. Linear topological spaces

A set X of elements x,y,z,... is said to be a linear topological space if the following conditions are fulfilled:

I. X is a linear space.

II. X is a topological space.

III. Addition and scalar multiplication are continuous in the topology of X.

In particular the continuity of the addition

$$x_o + y_o = z_o$$

means that for any neighbourhood W of z_o there are neighbourhoods U of x_o and V of y_o such that $U+V \subset W$.
In particular the continuity of the scalar multiplication

$$\lambda_o x_o = y_o$$

means that for any neighbourhood V of y_o there is an ε-neighbourhood of λ_o and a neighbourhood U of x_o such that for $|\lambda-\lambda_o|<\varepsilon$ $\lambda U \subset V$.

From the continuity of the addition it follows that the topology in X is completely determined by the neighbourhoods of zero. In fact, translation of the neighbourhoods U of zero by means of V=x+U determines a system of neighbourhoods V at x which by virtue of the continuity is equivalent to the original system at x, i.e. in every neighbourhood of the translated system there is a neighbourhood of the original system, and vice versa.

Therefore, the topology in a l.t.s. is completely determined by a base of zero-neighbourhoods.

From the continuity of the scalar multiplication it follows without difficulty that if U is open then also λU is open for all $\lambda \neq 0$. If U is a neighbourhood of zero a balanced neighbourhood may be formed by taking the union of all λU with $|\lambda| \leq \varepsilon$ for some positive ε.

Therefore any base of zero-neighbourhoods can be transformed into a base of balanced zero-neighbourhoods.

We shall now give an important example of a l.t.s.

Example

The space K(a) consists of all infinitely differentiable functions $\varphi(x)$ on the interval $(-\infty,\infty)$ which vanish outside the interval $(-a,a)$. The linear relations are the usual ones. The zero-neighbourhood $U(m,\varepsilon)$ is defined as the set of those $\varphi(x)$ for which

(3.1) $\max|\varphi(x)|<\varepsilon$, $\max|\varphi'(x)|<\varepsilon$, ..., $\max|\varphi^{(m)}(x)|<\varepsilon$;

the collection of zero-neighbourhoods consists of all $U(m,\varepsilon)$ with $m=0,1,2,\ldots$ and $\varepsilon>0$. It is easily seen that by this a topology in $K(a)$ is defined. Further it is clear that there is countable base at zero (take $\varepsilon=1, \frac{1}{2}, \frac{1}{3}, \ldots$) so that the space is separable. Convergence $\varphi_n(x)\longrightarrow\varphi(x)$ means that each derivative $\varphi_n^{(j)}(x)$ $(j=0,1,2,\ldots)$ tends uniformly to the limit $\varphi^{(j)}(x)$.

A set S in a l.t.s. is said to be bounded if it is absorbed by each neighbourhood U of zero, i.e. if for each neighbourhood U of zero there is a positive real λ for which $S\subset\lambda U$.

The set consisting of a single point x is bounded since $\lambda x\longrightarrow 0$ for $\lambda\longrightarrow 0$.

If the sets S_1 and S_2 are bounded then the same is true for their union and their arithmetical sum. If in particular S_2 consists of a single point it follows that translation does not destroy the property of boundedness.

A sequence of elements x_n which converges to a limit x forms a bounded set. In order to prove this we may assume $x=0$. Since any neigbourhood of zero contains a balanced neighbourhood it is sufficient to prove the absorbing property for an arbitrary balanced neighbourhood U of zero. Since for some N $x_n\in U$ for $n>N$ the proof can easily be completed.

The closure of a bounded set is also bounded. The proof depends on the property that any neighbourhood U of zero contains a smaller neighbourhood V such that $\overline{V}\subset U$. A topological space which possesses this property is said to be a regular space. The proof that any linear topological space is regular depends on the continuity of subtraction. In fact if U is an arbitrary neighbourhood of zero then the continuity of subtraction secures the existence of neighbourhoods V_1 and V_2 for which $V_1-V_2\subset U$. The intersection of V_1 and V_2 contains finally a neighbourhood W for which it is easy to show that $\overline{W}\subset U$.

Theorem 3.1 . A set S is bounded if and only if for every sequence $x_n\in S$ $(n=1,2,\ldots)$ the elements x_n/n converge to zero.

Proof. Elementary.

Next we show that in any linear topological space compactness always implies boundedness.

Theorem 3.2. In a linear topological space any compact subset is bounded.

Proof.

We shall show that a contradiction is obtained if $S \subset X$ is compact but not bounded. If S is not bounded there exists a sequence $x_n \in S$ $(n=1,2,\ldots)$ and a zero neighbourhood U such that $x_n/n \notin U$. Then the sequence x_n is itself not bounded and has no bounded subsequence. But then the sequence contains no converging subsequence in contradiction to the assumption of compactness.

The definition of a continuous linear operator A which transforms a l.t.s. X into a l.t.s. X_1 is implied in the definition of linearity in section 1 and that of continuity in section 2. We repeat:

The operator A is said to be linear and continuous if the following conditions are satisfied

1^o
$$A(\alpha x+\beta y) = \alpha Ax + \beta Ay.$$

2^o For any neighbourhood V of Ax there is a neighbourhood U of x the image of which is in V.

An operator which transforms any bounded set into a bounded set is said to be bounded. The following two theorems express the fact that in a space satisfying the first axiom of countability bounded linear operators and continuous linear operators are identical.

Theorem 3.3. Each continuous linear operator is bounded.

Proof.

Let A transform the bounded set S into a set S' and let x_n' be an arbitrary sequence of elements of S'. Then there is a sequence x_n in S such that $Ax_n=x_n'$. Since $x_n/n \to 0$ and $A(x_n/n)=x_n'/n$ also $x_n'/n \to 0$. Therefore S' is bounded.

Theorem 3.4. If in X the first axiom of countability is satisfied each bounded linear operator is continuous.

Proof.

If A is not continuous then there exists a zero neighbourhood V of X' and a shrinking base U_n $(n=1,2,\ldots)$ of X for which it is possible to find elements $x_n \in n^{-1}U_n$ with $Ax_n \notin V$. The sequence nx_n is bounded, and even converges to zero. However, it is easy to see that the transformed sequence $x_n'=nAx_n$ is not bounded.

From these two theorems the following conclusion may be drawn:

Theorem 3.5. If X satisfies the first axiom of countability a linear operator A is continuous if and only if $x_n \to 0$ implies $Ax_n \to 0$.

Application. In the space K(a) the operator of multiplication by x:

$$A\,\varphi(x) = x\,\varphi(x) = \psi(x)$$

is a continuous linear operator. Convergence $\varphi_n(x) \to 0$ means that $\varphi_n^{(j)}(x) \to 0$ uniformly in x for $j=0,1,2,\ldots$.
Since

$$\psi_n^{(j)}(x) = x\,\varphi_n^{(j)}(x) + j\,\varphi_n^{(j-1)}(x)$$

also $\psi_n(x) \to 0$ in K(a). Therefore A is continuous.

An operator f which transforms the elements x of a l.t.s. X into (real) numbers is called a (real) functional.

The functional (f,x) is said to be a continuous linear functional if

1°
$$(f, \alpha x + \beta y) = \alpha (f,x) + \beta (f,y).$$

2° For any $\varepsilon > 0$ there is a neighbourhood U of zero such that

$$|(f,x)| < \varepsilon \quad \text{for } x \in U.$$

This definition is clearly a specialization of that of a continuous linear operator.

Example

In the l.t.s. K(a) a functional is defined by

$$(f,\varphi) = \int_{-a}^{a} \varphi^{(m)}(x)\,d\,\mu(x), \tag{3.2}$$

where $\mu(x)$ is a function of bounded variation and where m is fixed. The functional is clearly linear. The continuity can be shown as follows. Take the zero neighbourhood

$$\max \left\{ |\varphi(x)|,\ |\varphi'(x)|, \ldots, |\varphi^{(m)}(x)| \right\} < \delta ,$$

then (3.2) gives

$$|(f,\varphi)| < \delta \text{ var } \mu(x)$$

from which the continuity follows at once.

A functional which is bounded on any bounded set is said to be bounded. Also this definition is a specialization of the corresponding one for an operator. Again in a space with the first axiom of countability a continuous linear functional is bounded and conversely. For future reference we merely state the following specializations of the theorems 3.3, 3.4, and 3.5.

Theorem 3.6. A continuous linear functional is bounded.

Theorem 3.7. In a space with the first axiom of countability a bounded linear functional is continuous.
Theorem 3.8. In a space with the first axiom of countability a linear functional (f,x) is continuous if and only if $x_n \to 0$ implies $(f,x_n) \to 0$.

For continuous linear functionals $f,g,\ldots$ on a l.t.s. X addition and scalar multiplication can be defined by

$$(\alpha f+\beta g,x) = \alpha(f,x) + \beta(g,x). \tag{3.3}$$

Hence the continuous linear functionals on X form themselves a linear space X' said to be the conjugate of X.

A topology in X' can be introduced by defining as neighbourhood of zero the sets $U(\varepsilon,x_1,x_2,\ldots,x_m)$ consisting of those f for which

$$|(f,x_1)| < \varepsilon \ , \quad |(f,x_2)| < \varepsilon \ ,\ldots, \quad |(f,x_m)| < \varepsilon \ , \tag{3.4}$$

where $x_1,x_2,\ldots,x_m$ are an arbitrary finite set of X.

The verification that X' satisfies the axioms of a l.t.s. is left to the reader.

The topology introduced in this way is called a weak topology. The topology of the conjugate space does not satisfy the first axioms of countability except in trivial cases.

Convergence in this type of topology is called weak convergence. Convergence of a sequence f_n to a limit f means here that for each element x of X we must have

$$(f_n,x) \to (f,x).$$

The weak topology in X' further allows the introduction of notions as weakly bounded etc.

A. A sequence f_n is called a weakly fundamental sequence if for each weak neighbourhood U of X' there exists a number N such that f_m-f_n belongs to U for each m and $n \geq N$. The conjugate space X' is called weakly complete if every weakly fundamental sequence has a limit in X'.

A different type of topology in X' can be introduced by defining a system of neighbourhoods of zero with respect to an arbitrary bounded set $B \subset X$ and a positive constant ε as follows. The neighbourhood of zero $U(B,\varepsilon)$ is the set of all elements $f \in X'$ satisfying

$$\sup |(f,x)| < \varepsilon \qquad \text{for all } x \in B.$$

The collection of neighbourhoods is then the set $U(B,\varepsilon)$ for which B is bounded and $\varepsilon > 0$.

The verification of the axioms of the l.t.s. may be left to the reader. The latter type of topology is called a strong topology. This topology may not satisfy the first axiom of countability either.

Convergence in this type of topology is called strong convergence. Convergence $f_n \to f$ in the strong sense means that

$$(f_n, x) \to (f, x) \text{ uniformly in every bounded set.}$$

B. A sequence f_n is called a strongly fundamental sequence if for each strong neighbourhood U of X' there exists a number N such that $f_m - f_n$ belongs to U for each m and $n \geq N$.
The conjugate space X' is called strongly complete if every strongly fundamental sequence has a limit in X'.

It is obvious that the strong topology always includes the weak topology so that e.g. a strongly convergent sequence is always weakly convergent. Conversely completeness in the weak topology implies completeness in the strong topology.

Theorem 3.9. If X satisfies the first axiom of countability the conjugate X' is complete in the strong sense.
Proof.
Let f_n be a strong fundamental sequence. Then for each $x \in X$ the sequence (f_n, x) defines a limit (f,x) which obviously represents a linear functional. There remains to show its continuity. Applying theorem 3.7 it is sufficient to show that (f,x) is bounded on any bounded set B. But on B the numbers (f_n,x) are bounded and uniformly convergent to (f,x). Hence the limit (f,x) is also bounded on B q.e.d.

According to the general definition the subset S of X' is bounded in the strong sense if for each (strong) zero neighbourhood there is a constant $\lambda > 0$ with $\lambda S \subset U$. However, a less formal definition may be given in the following way.
The subset S of X' is said to be bounded on the subset A of X if the numbers $|(f,x)|$ are bounded for $f \in S$, $x \in A$ i.e. if

$$\sup_{x \in A, f \in S} |(f,x)| < \infty . \tag{3.5}$$

We shall show that S is strongly bounded if and only if S is bounded on every bounded subset A of X.

Proof.

1^o. Let S be strongly bounded.

If A is a bounded subset of X a strong neighbourhood of zero $U(A,1)$ exists consisting of those f for which

$$\sup_{x \in A} |(f,x)| < 1.$$

The boundedness of S means that $\lambda S \subset U$ for some λ. Hence it follows that

$$|(f,x)| < 1/\lambda$$

for all $f \in S$ and $x \in A$.

2^o Let S be bounded on an arbitrary bounded set A. Let $U(A,\varepsilon)$ be a strong neighbourhood of zero, i.e. a set for which

$$\sup_{x \in A} |(f,x)| < \varepsilon ,$$

Since we know that S is bounded on A we have $|(f,x)| < C$ for $f \in S$ and $x \in A$. But this means $\lambda S \subset U$ for $\lambda = \varepsilon/C$.

We may prove that, at least for spaces satisfying the first axiom of countability, a strongly bounded set S is not only bounded on all bounded subsets $A \subset X$ but that S is also bounded on some neighbourhood of zero. We note that a neighbourhood of zero is not necessarily a bounded set.

Theorem 3.10. If X satisfies the first axiom of countability then any strongly bounded subset $S \subset X'$ is bounded on some zero neighbourhood of X.

Proof.

Let $U_1 \supset U_2 \supset \ldots$ be a base of shrinking zero neighbourhoods. If the lemma is not true then for each n $(n=1,2,\ldots)$ we may take an element $x_n \in U_n$ and a functional $f_n \in S$ such that $|(f_n,x_n)| > n$. Then the sequence x_n converges to zero in X and is therefore bounded. Because S is strongly bounded it is bounded on the set x_n and there is a constant C with $|(f,x_n)| < C$ for $f \in S$ and all n. Hence a contradiction is obtained.

Obviously the converse of this lemma is true irrespective of countability.

4. Linear metric spaces

A set X of elements x,y,z,... is said to be a metric space if to each pair of elements x,y there is associated a nonnegative real number d(x,y) called the distance and which satisfies the following conditions.

1° $d(x,y)=0$ if and only if $x=y$

2° $d(x,y)=d(y,x)$, (symmetry)

3° $d(x,y) + d(y,z) \geqq d(x,z)$, (triangle inequality).

A metric space is a particular case of a topological space. The open neighbourhoods at x_o are defined as the sets $U(x_o,\varepsilon)$ of those points for which

$$d(x,x_o) < \varepsilon \qquad (\varepsilon > 0).$$

The neighbourhood $U(x_o,\varepsilon)$ may be called an open sphere with the radius ε. At any point the neighbourhoods are determined by a countable base of nesting spheres (e.g. $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$). Convergence $x_n \to x$ in a metric space means that $d(x_n,x) \to 0$.

The following example shows that a metric space may not be separable.

Example

Consider the set of all bounded sequences $x=(x_1, x_2,\ldots)$ of real numbers. It is easily seen that a metric space is defined by

$$d(x,y)= \sup |x_j-y_j| \qquad (j=1,2,\ldots).$$

The subset of all sequences x with x_n either zero or one is certainly not countable (real numbers in (0,1)). The distance between any two elements of the subset is 1. If to each element of the subset we associate spheres of radius $\frac{1}{2}$ a not countable system of disjunct (spherical) neighbourhoods is obtained which excludes the possibility of a countable dense set; any dense set must have at least one member in each sphere.

A sequence x_n $(n=1,2,\ldots)$ of points of a metric space is said to be a fundamental sequence or a Cauchy sequence if for each $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that

$$d(x_m,x_n) < \varepsilon \qquad \text{for all } m,n > N(\varepsilon).$$

It follows directly from the triangle axiom that every convergent sequence is fundamental.

If every fundamental sequence in X converges to an element of X the space is said to be complete.

Example

The space of all continuous functions $f(x)$, $0 \leq x \leq 1$ with the metric

$$(4.1) \qquad d(f,g) = \max |f(x) - g(x)|$$

is a complete metric space.

A complete metric space X^* is said to be the completion of the (not necessarily complete) metric space X if X is a subspace of X^* which is dense in X^*.

Every metric space can be embedded in a complete space by a process of completion which is analogous to the extension of the class of the rational numbers to that of the real numbers.

Theorem 4.1 (Baire)

If a complete metric space X is the countable sum of sets S_n $(n=1,2,\dots)$ then at least one of these sets is dense in some sphere.

Proof

We may assume that $S_1 \subset S_2 \subset S_3 \subset \dots$. If the proposition were not true a contradiction would be obtained in the following way. Take an arbitrary sphere R_0 ; in it there exists another sphere R_1 which contains no points of S_1 ; in R_1 there exists a sphere R_2 which contains no points of S_2 etc.

We obtain a sequence of nesting spheres R_n the radius of which may be chosen $< 1/2^n$ say. It is easily seen that the centres form a fundamental sequence which has a limit $x \in X$ in view of the completeness of X.

However, by the construction x belongs to no S_n. Since X is the union of all S_n a contradiction is obtained.

The latter theorem may also be put in the equivalent forms

Theorem 4.1^a

If a complete metric space X is the countable sum of closed sets S_n then at least one of these sets contains an open set.

Theorem 4.1^b

It is impossible to obtain a complete metric space as the countable sum of nowhere dense sets.

By a nowhere dense set is obviously meant a set S in X such that

the closure $\bar{S}$ does not contain an open set.
Baire's theorem may be considered as an existence theorem of the same kind as e.g. the principle of Bolzano-Weierstrass. Many important theorems such as Banach's theorem of the inverse operator and the Banach-Steinhaus theorem on the principle of uniform boundedness depend ultimately upon Baire's theorem.

The following related existence theorem demonstrates the applicability of Baire's principle.

Theorem 4.2

Let S be a convex central-symmetric closed absorbing set in a complete linear metric space X, then S contains a neighbourhood of zero.

Proof

The union of all sets nS $(n=1,2,\ldots)$ gives all elements of X. Hence according to Baire's theorem in the version of theorem 4.1^a some set mS contains a sphere. Then the same must be true for S. The remainder of the proof follows geometrical intuition.
For let S contain the sphere (x_0, ρ) then S also contains the opposite sphere $(-x_0, \rho)$. Since S is also convex it contains the cylindrical region determined by these spheres, i.e. the convex hull. Then S finally contains the sphere $(0, \rho)$.

We recall the definition of compactness. A set S in a metric space X is compact if every sequence in S contains a subsequence which converges to some element of X. The concept of compactness is closely related to that of total boundedness which will be introduced now.

The set R is said to be an ε-net with respect to the set S in a metric space X if for any point $x \in S$ there is at least one point $a \in R$ such that

$$d(a,x) < \varepsilon .$$

The set S is said to be totally bounded if X contains a finite ε-net with respect to S for every positive ε.

It is obvious that a totally bounded set is bounded i.e. is contained in some sphere. However, the converse need not be true. Further, if S is totally bounded then also its closure $\bar{S}$ is totally bounded.
In particular S may be the whole space X. If X is totally bounded it is separable. In fact, construct a sequence of $(\varepsilon = 1/n)$ nets. The union is obviously dense in X.

<u>Theorem 4.3</u> (Hausdorff)
In a metric space X any compact set is totally bounded. If, moreover, X is complete any totally bounded set is compact.
<u>Proof</u>
<u>1.</u> Let S be a compact set.
If X contains no ε-net for S it is possible to construct a sequence $x_n \in S$ $(n=1,2,\dots)$ such that $d(x_i,x_j) \geq \varepsilon$ for any pair i,j.
However, this means that x_n does not converge and does not contain a converging subsequence. This contradicts the assumption of compactness.
<u>2.</u> Let S be a totally bounded set in a complete metric space. We consider a sequence R_n of ε_n-nets with $\varepsilon_n = 1/n$ $(n=1,2,\dots)$. If s is a given infinite sequence of points x_n of S then R_1 contains a point r_1, such that a sphere of radius 1 about r_1 contains an infinite subsequence s_1 of s. Next R_2 contains a point r_2 such that a sphere of radius 1/2 about r_2 contains an infinite subsequence s_2 of s_1. Continuing this process a sequence of sequences $s_1 \supset s_2 \supset s_3 \supset \dots$ is obtained, say

$$\begin{array}{lllll} s_1 & x_{11} & x_{12} & x_{13} & \dots \\ s_2 & x_{21} & x_{22} & x_{23} & \dots \\ s_3 & x_{31} & x_{32} & x_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

The diagonal subsequence $x_{11}, x_{22}, x_{33} \dots$ is fundamental since all x_{nn} for $n \geq N$ belong to a sphere of radius 1/N, and converges to a limit in S in view of the completeness of S.
Therefore S is (sequentially) compact.

In analysis one of the more important metric spaces is the space C(0,1) of all continuous functions f(x), $0 \leq x \leq 1$. The metric of that space has been given above by (4.1). There is an important theorem concerning the compactness of a subset in this space which is due to Arzelà and Ascoli. First we need a definition.

A set of functions is said to be <u>equicontinuous</u> for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all functions f(x) of this set

$$|f(x_1) - f(x_2)| < \varepsilon$$

whenever $|x_1-x_2| < \delta(\varepsilon)$

Theorem 4.4 (Arzelà)

A set of continuous functions in C(0,1) is compact if and only if it is equicontinuous and uniformly bounded.

Proof

1. Let S be a compact subset.

We know already that S is bounded i.e. uniformly bounded. The preceding theorem says that there exists a finite 1/3 ε-net consisting of $g_1, g_2, \dots, g_m$.

Each function $g_k(x)$ $(k=1,2,\dots,m)$ is uniformly continuous so that there is a δ_k such that

$$|g_k(x') - g_k(x'')| < \frac{1}{3}\varepsilon \qquad \text{for } |x'-x''| < \delta_k.$$

Put $\delta = \min \delta_k$ then for $|x'-x''| < \delta$ we have

$$|f(x') - f(x'')| \leqq |f(x')-g_k(x')| + |f(x'') - g_k(x'')| +$$

$$+|g_k(x') - g_k(x'')| < \varepsilon$$

(with a suitable auxiliary g_k).

2. Let S be equicontinuous and uniformly bounded.

The rectangle $0 \leqq x \leqq 1$, $-M \leqq y \leqq M$, where M is the upper bound of $|f(x)|$, is subdivided into cells with horizontal sides of length $< \delta$ where δ is such that $|f(x')-f(x'')| < \frac{1}{5}\varepsilon$ if $|x'-x''| < \delta$ simultaneously for all $f \in S$. To every function $f(x) \in C(0,1)$ we assign a polygonal arc $p(x)$ with vertices at the mesh points and such that

$$|f(x_k) - p(x_k)| < \frac{1}{5}\varepsilon .$$

Therefore

$$|p(x_k)-p(x_{k+1})| \leqq |p(x_k)-f(x_k)| + |p(x_{k+1}) - f(x_{k+1})| +$$

$$+ |f(x_k)-f(x_{k+1})| < \frac{3}{5}\varepsilon .$$

Since $p(x)$ is linear between x_k and x_{k+1}, we have

$$|p(x)-p(x_k)| < \frac{3}{5}\varepsilon \qquad \text{for } x_k \leqq x \leqq x_{k+1}.$$

Let x be an arbitrary point and x_k be the subdivision point which is closest to x from the left. Then

$$|f(x)-p(x)| \leqq |f(x)-f(x_k)| + |f(x_k)-p(x_k)| + |p(x_k)-p(x)| < \varepsilon .$$

Hence the finitely manypolygonal arcs $p(x)$ constructed above form an ε-net. Hence S is totally bounded.

Remark. The second part of this theorem is sometimes stated as follows: an infinite set of equicontinuous and uniformly bounded functions on $[0,1]$ contains a uniformly convergent subsequence.

A linear space X where a distance $d(x,y)$ is defined which satisfies the three axioms of a metric space is said to be a linear metric space if also the following axioms are satisfied:

4° $d(x,y) = d(x-y,0)$, (invariance of the metric).

5° $\lambda_n \to 0$ implies $d(\lambda_n x, 0) \to 0$ for arbitrary x.

6° $d(x_n,0) \to 0$ implies $d(\lambda x_n, 0) \to 0$ for arbitrary λ.

The fourth axiom says that $d(x+z, y+z)$ does not depend upon z. A simple consequence is

(4.2) $$d(-x,0) = d(x,0)$$

Another corollary is

(4.3) $$d(x+y,0) \leq d(x,0) + d(y,0).$$

A linear metric space is a particular case of a linear topological space with the usual topology of a metric space. The verification of the groups I and II of axioms of a l.t.s. is obvious. The verification of the continuity axioms III is not much more complicated.

The continuity of the addition $x_0+y_0=z_0$ follows from

$$d(x+y,z_0) \leq d(x+y,x+y_0) + d(x+y_0,\ x_0+y_0) =$$
$$= d(y,y_0) + d(x,x_0).$$

The continuity of the multiplication $\lambda_0 x_0 = y_0$ follows from

$$d(\lambda x, y_0) \leq d(\lambda x, \lambda x_0) + d(\lambda x_0, \lambda_0 x_0) =$$
$$= d(\lambda(x-x_0),\ 0) + d((\lambda-\lambda_0)x_0,\ 0).$$

and the axioms 5° and 6°

A set S in a linear metric space is bounded if for each sphere $d(x,0) < \varepsilon$ there exists a positive real λ for which λS is contained in the sphere.

Since a linear metric space satisfies the first axiom of countability the theorems 3.3. and 3.4 may be combined into the following.

Theorem 4.5

A linear operator which transforms a linear metric space into a linear metric space is continuous if and only if it transforms bounded sets into bounded sets.

In a linear metric space for each element x a number $|x|$ can be introduced by means of

$$(4.4) \qquad |x| \overset{\text{def}}{=} d(x,0).$$

It is easily seen that this number satisfies the following conditions

1° $|x| = 0$ if and only if $x=0$.

2° $|-x| = |x|$.

3° $|x+y| \leq |x| + |y|$.

In case $|\lambda x| = |\lambda||x|$ for each λ and x, $|x|$ is called the norm of x and the linear metric space is called a normed space. Such spaces will be studied in the following section.

The proof of the following important theorem due to Banach (1929) involves some subtle reasoning.

Theorem 4.6 A continuous linear operator which transforms a complete linear metric space into a similar space and for which there exists an inverse has a continuous inverse.

Before proving the theorem the following remarks will be made. Continuity of the operator T which maps the linear metric space X into the linear metric space Y means that for every sphere $S(|y| < \rho)$ in Y there exists a sphere $U(|x| < \varepsilon)$ in X such that $TU \subset S$. The continuity of the inverse T^{-1} means that for every sphere $U(|x| < \varepsilon)$ in X the image TU contains a sphere $S(|y| < \rho)$ in Y.

The proof of the theorem will consist of two parts. In the first part it will be shown that for any $U(|x| < \varepsilon)$ the image TU is dense in some sphere $S(|y| < \rho)$ i.e. that $\overline{TU} \supset S$. The proof of this part uses Baire's theorem 4.1. In the second part it will be shown by some limit process that TU contains a sphere S.

Proof.

If U is a given (spherical) neighbourhood of zero in X the continuity of subtraction secures the existence of a zero neighbourhood

V for which $V-V \subset U$. The union of the sets nV $(n=1,2,\dots)$ gives all elements of X. Hence, according to theorem 4.1 there is an index m for which TmV is dense in some sphere of Y. Then the same must be true for TV. Let $\overline{TV}$ contain the sphere W then $T(V-V)$ contains the open set $W-W$. The latter set evidently contains zero and therefore it contains a (spherical) neighbourhood S. Then S is also contained in $\overline{TU}$.

Consider next a sequence of spheres U_n $(|x| < 2^{-n}\varepsilon)$ $(n=1,2,\dots)$ in X and assume that $\overline{TU_n}$ contains a sphere $S_n(|y| < \rho_n)$ $(n=1,2,\dots)$ in Y, where $\rho_n \to 0$. We choose an arbitrary element y_o of S_1 and construct a sequence y_n $(n=1,2,\dots)$ of elements by taking y_n from TU_n and requiring that $|y_o-y_1| < \rho_2$, $|y_o-y_1-y_2| < \rho_3$, etc.

The possibility of the construction is implied in the fact that TU_n is dense in S_n. In this way the element y_o is expressed by a converging series

$$y_o = y_1+y_2+y_3+ \dots$$

If $x_1, x_2 \dots$ are the originals in X of the elements $y_1, y_2, \dots$ of Y it can easily be shown that there is an element $x_o \in X$ for which

$$x_o = x_1+x_2+x_3+ \dots$$

Since X is complete it is sufficient to prove the Cauchy criterion. In fact

$$|x_m + x_{m+1} + \dots + x_{m+p}| \leq 2^{-m}\varepsilon + 2^{-m-1}\varepsilon + \dots + 2^{-m-p}\varepsilon < 2^{-m+1}\varepsilon \to 0.$$

Further it follows that $|x_o| < \varepsilon$. The continuity of the operator T tells us that $y_o = Tx_o$. Since y_o was an arbitrary element of the sphere $S_1(|y| < \rho_1)$ it has been shown that the sphere $U(|x| < \varepsilon)$ has an image which contains the sphere $S_1(|y| < \rho_1)$, which proves the theorem.

5. Normed linear spaces

A linear space X of elements x,y,z,... with (real) scalars $\alpha, \beta, \gamma, \ldots$ is said to be a normed space or, more completely, a normed linear space if to each element x there is associated a nonnegative real number $\|x\|$ called the norm of x which satisfies the conditions.

$\underline{1}^{o}$ $\|x\| = 0$ if and only if $x=0$.

$\underline{2}^{o}$ $\|\alpha x\| = |\alpha| \, \|x\|$.

$\underline{3}^{o}$ $\|x+y\| \leqq \|x\| + \|y\|$.

It is easy to see that the norm is continuous since condition $\underline{3}^{o}$ yields $\big| \|x\| - \|y\| \big| \leqq \|x-y\|$.

These properties show that the distance function

$$d(x,y) \overset{\text{def}}{=} \|x-y\|$$

is an invariant metric of X. It can be verified without difficulty that this distance satisfies the six conditions of a linear metric space. The number $|x|$ which was introduced for a linear metric space in (4.4) clearly coincides with that of the normed space.
A normed space is distinguished by the fact that its norm satisfies the multiplicative property $\|\alpha x\| = |\alpha| \, \|x\|$, a property which is not enjoyed by other linear metric spaces.

The topology of a normed space is that of a linear metric space i.e. the neighbourhoods of zero are determined by spheres $U(\varepsilon)$ with $\|x\| < \varepsilon$. This topology satisfies the first axiom of countability. It is sometimes called norm topology or strong topology. A complete normed space is said to be a Banach space.

Example The space $C(0,1)$ of continuous functions $f(x)$ with the usual operations of addition and multiplication by a scalar with the norm

$$\|f(x)\| = \max |f(x)|$$

is a normed linear space. Convergence with respect to this norm is the usual uniform convergence. Hence the space is a Banach space.

The second axiom of the norm, a property which is not satisfied by the norm of a linear metric space, makes it possible to state the boundedness of a set in a very simple way and it has further a number of important consequences e.g. in connection with the properties of the conjugate space.

A set S in a normed linear space is bounded if and only if there exists a constant C such that $\|x\| < C$ for all $x \in S$.

In a normed linear space a continuous linear operator A which transforms the n.l.s. X into a n.l.s. X_1 is characterized by the following conditions which can easily be derived from the corresponding conditions in a linear topological space:

1^0 $\quad A(\alpha x + \beta y) = \alpha Ax + \beta Ay$

2^0 $\quad x_n \to 0 \quad$ implies $\quad Ax_n \to 0$

for any sequence x_n $(n=1,2,\ldots)$ of X.

The theorem 3.3 and 3.4 can be put in the following simplified form.

Theorem 5.1^a

A continuous linear operator $A(X \to X_1)$ transforms the unit sphere of X into a bounded set of X_1.

Theorem 5.1^b

A linear operator $A(X \to X_1)$ which transforms the unit sphere of X into a bounded set of X_1 is continuous.

These theorems may be put in a still simpler form if the following definition is introduced.

An operator $A(X \to X_1)$ is said to be bounded if there is a constant M such that

$$(5.1) \qquad \|Ax\| \leq M \|x\| \qquad \text{for all } x \in X.$$

Then the two theorems may be combined into

Theorem 5.1

For a linear operator boundedness and continuity are equivalent.

The greatest lower bound of the number M which satisfy the inequality (5.1) of a bounded linear operator A is called the norm of the operator and is written as $\|A\|$. Without difficulty it can be shown that

$$(5.2) \qquad \|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

In section 1 we have seen that the set of all linear operators $A(X \to X_1)$ is a linear space. The set of all bounded linear operators

is also a linear space, for the boundedness of A and B implies the boundedness of all linear combinations.
With the definition (5.2) of the norm of an operator the latter space becomes itself even a normed space. It is not difficult to verify that (5.2) satisfies the three axioms of the norm.

The following theorem can now easily be proved.

<u>Theorem 5.2</u>

The space of all bounded linear operators which transform a normed space into a Banach space is itself a Banach space.

<u>Proof</u>

Let A_n $(n=1,2,\ldots)$ be a Cauchy sequence of bounded linear operators which map X into X_1, then for each $x \in X$ we have
$\| A_m x - A_n x \| \leq \| A_m - A_n \| \; \| x \| \rightarrow 0.$

Therefore $A_n x$ is a Cauchy sequence in X_1 and this has a limit y since X_1 is complete.
The operator A defined by $Ax=y$ is the limit of A_n; and
$\| A_x \| = \lim \| A_n x \| \leq \lim \; \| A_n \| \; \| x \|$. Thus the operator space is complete.

We consider a set S of bounded linear operators A on a Banach space X. This set is said to be <u>weakly bounded</u> if for each $x \in X$ there is a constant C such that

(5.3) $\| A\, x \| < C$ for all $A \in S$.

We have the important result due to Banach and Steinhaus that a weakly bounded set is also uniformly bounded in the sense of (5.1). This property is known as the <u>principle of uniform boundedness</u>.

<u>Theorem 5.3</u> (Banach, Steinhaus)

If the bounded linear operators A on a Banach space are weakly bounded then their norms are uniformly bounded.

<u>Proof</u>

The boundedness of the set S of operators A in the weak sense means that $\sup_{A \in S} \| Ax \| < \infty$ for each $x \in X$. Let X_k $(k=1,2,\ldots)$ be the subset of X for which $\sup \| Ax \| \leq k$ uniformly in S. Since all A are continuous the subset X_k is closed. The union of all X_k gives X so that we may apply Baire's theorem 4.1. Hence there exists an index m for which X_m contains a closed sphere

$G(x_0,\delta)$. For $x \in G$ we have $\| A x \| \leq m$. Applying the triangle axiom we obtain $\| A x \| < 2m$ for $\| x \| < \delta$. This means $\| A \| \leq 2m / \delta$ uniformly in S.

We shall also reformulate the important theorem of Banach 4.5 on the inverse operator on a Banach space.

Theorem 5.4 (Banach)
The inverse of a linear continuous operator which transforms a Banach space one-to-one onto a Banach space is continuous.

A number of properties of linear functionals and the conjugate space formed by them have been discussed from a general point of view in section 3. For linear functionals defined on a normed space a number of simplifications may be introduced. We shall briefly repeat the main notions.

A functional (f,x) is continuous if $x_n \to 0$ implies $(f,x_n) \to 0$.

A functional (f,x) is bounded if there exists a constant M such that

$$(5.4) \quad |(f,x)| \leq M \|x\| \qquad \text{for all} \quad x \in X.$$

From theorem 5.1 it follows that boundedness and continuity are equivalent.

The greatest lower bound of M is called the norm of the functional f and is written as $\| f \|$. Since this norm is a special case of that of a bounded operator we have the following equivalent of (5.3)

$$(5.5) \quad \| f \| = \sup_{\|x\|=1} |(f,x)| = \sup_{x \neq 0} \frac{|(f,x)|}{\| x \|}.$$

In section 3 we introduced two different topologies in the space X' of linear continuous functionals.

In the weak topology of X' the neighbourhoods of zero $U(\varepsilon, x_1, x_2, \ldots, x_m)$ are defined as the sets for which

$$(5.6) \quad |(f,x_1)| < \varepsilon, \quad |(f,x_2)| < \varepsilon, \quad \ldots, \quad |(f,x_m)| < \varepsilon.$$

A set S of X' is weakly bounded if for each $x \in X$ there is a constant C such that

$$|(f,x)| < C \qquad \text{for all } f \in S$$

A sequence f_n $(n=1,2,\ldots)$ is weakly convergent to the limit f if

$$(f_n, x) \rightarrow (f, x) \qquad \text{for each } x \in X.$$

The strong topology of X' may be formulated in a much simpler way than in the general case of a linear topological space. In the latter case the neighbourhoods of zero had to be defined with reference to an arbitrary bounded set $B \subset X$.
Here for a normed space X we may restrict ourselves to one single bounded set viz. the unit sphere $\|x\| = 1$.

In the strong topology of X' the neighbourhoods of zero $U(\varepsilon)$ are defined as the spheres

$$\|f\| < \varepsilon . \tag{5.7}$$

A set S is strongly bounded if there is a constant C such that

$$\|f\| < C \qquad \text{for all } f \in S.$$

A sequence f_n $(n=1,2,\ldots)$ is strongly convergent to the limit f if

$$\|f_n - f\| \rightarrow 0.$$

It will be clear that strong convergence implies weak convergence.

With respect to the strong topology we have the following particular case of theorem 5.2. (see also theorem 3.9).

Theorem 5.5

The conjugate of a normed space is a Banach space.

Proof

A functional maps a normed space into the Banach space of real numbers.

We come now to the discussion of a very important theorem due to Hahn and Banach concerning the extension of a continuous linear functional.

Theorem 5.6 (Hahn-Banach)

Every continuous linear functional f which is defined on a linear subspace M of a normed space X can be extended to the entire space with preservation of the norm.

Proof

The theorem will be explicitly proved for a separable space only. For non-separable spaces the proof is essentially the same, but requires the so-called axiom of choice; we do not discuss this, since we need the result only for separable spaces.

First, we shall extend the functional to the linear subspace M_1 obtained by adding to M some element $x_0 \notin M$. An arbitrary element y of M_1 is uniquely representable as

$$y = x - tx_0 , \quad x \in M$$

For the extension F of f upon M_1 we must have

$$(F,y) = (f,x) - ct$$

where $c = (F,x_0)$. The problem consists in finding a value of c which keeps the norm intact,
This means that

$$|(f,x) - ct| \leq \|f\| \, \|x - tx_0\| \qquad \text{for all } x \in M.$$

We may safely assume that $t \neq 0$. Then by putting x=zt the last inequality can be replaced by

$$|(f,z) - c| \leq \|f\| \, \|z-x_0\| \qquad \text{for all } z \in M,$$

or, what amounts to the same thing

$$(f,z) - \|f\| \, \|z-x_0\| \leq c \leq (f,z) + \|f\| \, \|z-x_0\|$$

for all $z \in M$.
The possibility of satisfying this inequality for all z in M follows from the following consideration. For any pair $z_1, z_2 \in M$ we have

$$(f,z_1)-(f,z_2) \leq \|f\| \|z_1-z_2\| \leq \|f\| \{ \|z_1-x_0\| + \|z_2-x_0\| \} ,$$

so that

$$(f,z_1) - \|f\| \, \|z_1-x_0\| \leq (f,z_2) + \|f\| \, \|z_2-x_0\| .$$

If c_1 is the least upper bound of the left-hand side and c_2 the greatest lower bound of the right-hand side then the choice $c_1 \leq c \leq c_2$ in fact guarantees the preservation of the norm.

The separability of X implies the existence of a countable dense set $x_1, x_2 \ldots$. Then the preceding procedure may be carried out for each x_n (n=1,2,...). Thus we obtain a functional F on a set which is everywhere dense in X. At the remaining points of X the functional is defined by continuity.

The theorem of Hahn-Banach guarantees for any normed space the existence of at least one non-trivial continuous linear functional. In fact we have the following simple corollary.

Corollary

For every normed space X and any fixed element $x_o \in X$ there exists a continuous linear functional (f,x) with the properties

a $\|f\| = 1$. b $(f,x_o) = \|x_o\|$.

The Banach space X with elements x and its conjugate X' with elements f define for each pair $x \in X$, $f \in X'$ a scalar value (f,x). By this not only a functional f on X is defined but also a functional, say ξ, on X'. The latter functional is linear and continuous. We may prove that its norm defined as (5.5) coincides with $\|x\|$. This is expressed by

Theorem 5.7

$$(5.8) \qquad \sup_{f \in X'} \frac{|(f,x)|}{\|f\|} = \|x\| .$$

Proof

It follows from (5.5) that $|(f,x)| / \|f\| \leq \|x\|$ so that $\|\xi\| \leq \|x\|$.

According to the previous corollary a continuous linear functional f_o can be defined with the properties $\|f_o\| = 1$ and $|(f_o,x)| = \|x\|$. Taking this special element of X' the equality sign is obtained.

The following two theorems are known under the "principle of uniform boundedness". The first theorem is just a specialization of theorem 5.3 of Banach-Steinhaus for functionals. The second theorem is a version in which the role of X and its conjugate are interchanged. The latter theorem uses theorem 5.7, so that it depends on the Hahn-Banach principle.

Theorem 5.8

Each weakly bounded set of continuous linear functionals on a Banach space is also strongly bounded.

This theorem may be stated also as

$$(5.9) \qquad \sup_{f \in S'} |(f,x)| < \infty \qquad \text{implies} \quad \sup \|f\| < \infty .$$

Corollary 1

If f_n is a weakly convergent sequence of continuous linear functionals on a Banach space then the sequence $\|f_n\|$ is uniformly bounded.

The latter corollary leads to the following criterion for weak convergence of continuous linear functionals:

Corollary 2

The sequence f_n of continuous linear functionals on a Banach space is weakly convergent to zero if and only if the norms $\| f_n \|$ $(n=1,2,\dots)$ are uniformly bounded and $(f_n,x)\to 0$ at least for those x belonging to a set which is dense in X.

Theorem 5.9

If every continuous linear functional on a linear normed space X is bounded on a subset S of X then S is bounded. Symbolically

$$\sup_{x\in S} |(f,x)| < \infty \qquad \text{implies} \qquad \sup \|x\| < \infty$$

Proof

Follows easily from the theorems 5.7 and 5.8

In the theory of distributions we need some knowledge of the continuous linear functionals on the space of the continuous functions $\varphi(x)$ in a finite interval, say $0 \leq x \leq 1$. We have the following important theorem due to F.Riesz (1909) called the Riesz representation theorem:

Theorem 5.10 (Riesz)

Every continuous linear functional (f,φ) on the space $C(0,1)$ of continuous functions is of the form

$$(f,\varphi) = \int_0^1 \varphi(x)\, d\,\mu(x),$$

where $\mu(x)$ is of bounded variation.

Proof

The space $C(0,1)$ may be considered as a subspace of the space $D(0,1)$ of all piecewise continuous functions $\psi(x)$, i.e. of functions $\psi(x)$ which have a finite number of ordinary jumps. Both spaces are normed spaces with the usual norm

$$\|\psi\| = \sup \psi(x) .$$

The functional f on C can be extended in D by means of the theorem of Hahn-Banach. If F is the extended functional we may define a function $\mu(t)$ by means of

$$(F, \ \theta(t-x)) = \mu(t) \ ,$$

where $\theta(x)$ is the unit-step function

$$\theta(x)=0 \quad x \leqq 0 \ , \qquad \theta(x)=1 \qquad x > 0 \ .$$

It can easily be shown that $\mu(t)$ is of bounded variation. In fact, divide (0,1) in

$$t_o=0 < t_1 < t_2 < \ldots < t_{n-1} < t_n=1$$

Then we have

$$\sum_{j=1}^{n} | g(t_j) - g(t_{j-1}) | = \sum_{j=1}^{n} |(F, \ \theta(t_j-x) - \theta(t_{j-1}-x))| =$$

$$= (F, \sum_{j=1}^{n} \pm \left\{ \theta(t_j-x) - \theta(t_{j-1}-x) \right\}) \leqq$$

$$\leqq \| F \| \ | \sum_{j=1}^{n} \pm \left\{ \theta(t_j-x) - \theta(t_{j-1}-x) \right\} | \leqq \| F \| = \| f \| < \infty \ .$$

Any continuous function $\varphi(x)$ may be approximated uniformly by the sequence

$$\varphi_n(x) = \sum_{k=1}^{n} \varphi(\tfrac{k}{n}) \left\{ \theta\left(\tfrac{k}{n}-x\right) - \theta\left(\tfrac{k-1}{n} - x\right) \right\} .$$

Then

$$(F, \varphi) = \lim \, (F, \varphi_n) = \lim \sum_{k=1}^{n} \varphi\left(\tfrac{k}{n}\right) \left\{ \mu\left(\tfrac{k}{n}\right) - \mu\left(\tfrac{k-1}{n}\right) \right\} =$$

$$= \int_0^1 \varphi(t) \ d \ \mu(t).$$